# Lecture 3



Just as above, we can combine the parametric equations:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

$$\downarrow a \neq 0 \qquad \downarrow b \neq 0 \qquad \downarrow c \neq 0$$

$$t = \frac{x - x_0}{a} \qquad t = \frac{y - y_0}{b} \qquad t = \frac{z - z_0}{c}$$

Combining these together, we get the

$$\boxed{\text{Symmetric Equations of a line} : \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

It could happen that one (or even 2) of the components of $\vec{v}$ could be zero. An example is if $a = 0$, then the symmetric equations would take the form: $\boxed{x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}}$

Ex: Find the symmetric equations of the line in the previous example.

Sol: $\dfrac{x-(-2)}{3} = \dfrac{y-4}{-3} = \dfrac{z-0}{1}$

$$\Rightarrow \quad \frac{x+2}{3} = -\left(\frac{y-4}{3}\right) = z$$

$\square$

Sometimes, we don't want a whole line, but just a line segment. If we already have an equation for the whole line, we can just restrict the parameter $t$ to start at the first point and end at the second. So, you end up with something like this:

$$\vec{\ell}(t) = \vec{P}_0 + t\vec{v}, \quad a \le t \le b.$$

The quickest way to parametrize a line segment, however, is as follows:

If we want the line segment from $P$ to $Q$, it's parametrized by:

$$\boxed{\vec{\ell}(t) = (1-t)\overrightarrow{OP} + t\,\overrightarrow{OQ}, \quad 0 \le t \le 1}$$

In the plane, we know two lines are either parallel or they intersect. Lines in space, however, can be both non-parallel <u>AND</u> non-intersecting. These are called <u>skew</u> lines.

<u>Ex</u>: Show that the lines:

$$L_1: x = 3+2t,\ y = 4-t,\ z = 1+3t$$

$$L_2: x = 1+4s,\ y = 3-2s,\ z = 4+5s$$

are skew.

<u>Sol</u>: This is done in 2 steps. First, we show they're not parallel. This is as easy as checking if their direction vectors are parallel. The direction vectors are: $\vec{v}_1 = \langle 2,-1,3\rangle$ for $L_1$ and $\vec{v}_2 = \langle 4,-2,5\rangle$ for $L_2$. It's easy to see that one is not a multiple of the other, so the lines are not parallel. To see if the lines intersect, we set them equal to each other and try to solve the system:

$$\begin{cases} 3+2t = 1+4s \\ 4-t = 3-2s \\ 1+3t = 4+5s \end{cases}$$

$$\Longrightarrow \begin{cases} 2t-4s = -2 & (1) \\ -t+2s = -1 & (2) \\ 3t-5s = 4 & (3) \end{cases}$$

(1) $\Rightarrow t-2s=-1$ & (2) $\Rightarrow t-2s=1$

These contradict each other (you can't equal 1 AND -1 at the same time), so the system has no solution. So, the lines do not intersect. Thus, the lines are skew.

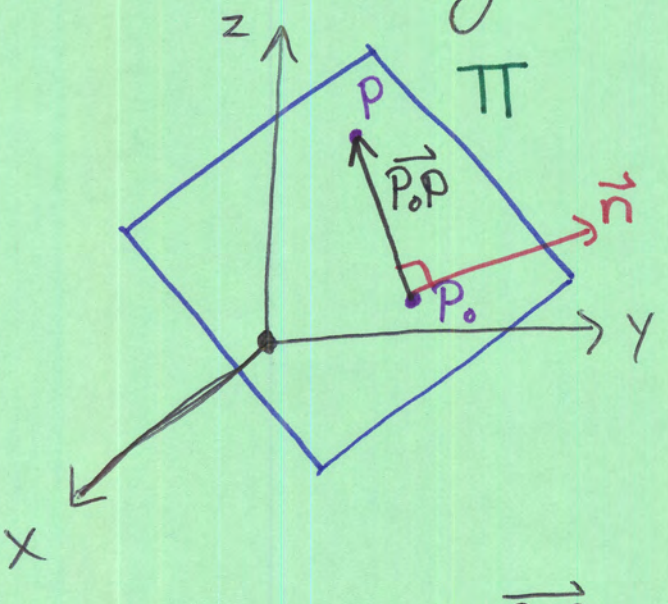
The natural generalization of a line is a plane. We again need two pieces of information to get the equation of a plane:

① A point $P_0 = (x_0, y_0, z_0)$ in the plane

② A vector normal (perpendicular) to the plane

$$\vec{n} = \langle a, b, c \rangle$$

How does this give us a plane?




$P = (x, y, z)$ is any point in the plane

Notice how $\vec{n} \perp \overrightarrow{P_0P}$ for any point $P$ in the plane. So, an equation for the plane is

| | |
|---|---|
| Vector equation of the plane Π | $\vec{n} \cdot \overrightarrow{P_0P} = 0$ |

Filling in $\vec{n} = \langle a,b,c\rangle$ & $\overrightarrow{P_0P} = \langle x-x_0, y-y_0, z-z_0\rangle$ gives the <u>scalar equation of the plane</u>:

$$\vec{n}\cdot\overrightarrow{P_0P} = \langle a,b,c\rangle\cdot\langle x-x_0, y-y_0, z-z_0\rangle$$
$$= \boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

Sometimes this is written as

$$ax+by+cz+d=0$$

where $d=-(ax_0+by_0+cz_0)$

<u>Ex</u>: Find an equation for the plane passing through $P=(0,1,1)$, $Q=(1,0,1)$, and $R=(1,1,0)$.

<u>Sol</u>: We already have a point in the plane (3 even!), so we just need the normal vector. notice we can make two vectors in the plane starting from P: $\overrightarrow{PQ}$ & $\overrightarrow{PR}$

$$\overrightarrow{PQ} = \langle 1-0, 0-1, 1-1\rangle = \langle 1,-1,0\rangle$$
$$\overrightarrow{PR} = \langle 1-0, 1-1, 0-1\rangle = \langle 1,0,-1\rangle$$

Now, we can use these two vectors in the plane (which are not parallel!) to make a normal vector by taking their cross product:

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

So, an equation is:

$$\vec{n} \cdot \langle x-0, y-1, z-1 \rangle = \langle 1,1,1 \rangle \cdot \langle x, y-1, z-1 \rangle$$
$$= x + (y-1) + (z-1) = 0$$

or, simplified $\quad x+y+z=2$

◇

Now, we have two kinds of objects in space: lines and planes. We already know the situation for two lines (intersecting, parallel, or skew), so how about the other pairs? Let's start with a line and a plane. Two things can happen: they're parallel or they intersect.

Ex: Does the line

$$L: x = 3+3t, \ y = t, \ z = -2+4t$$

intersect the plane $x+y+z=2$? If so, where?

Sol: If the line intersects the plane, we can plug the line into the equation for the plane and solve for a $t$ value.:

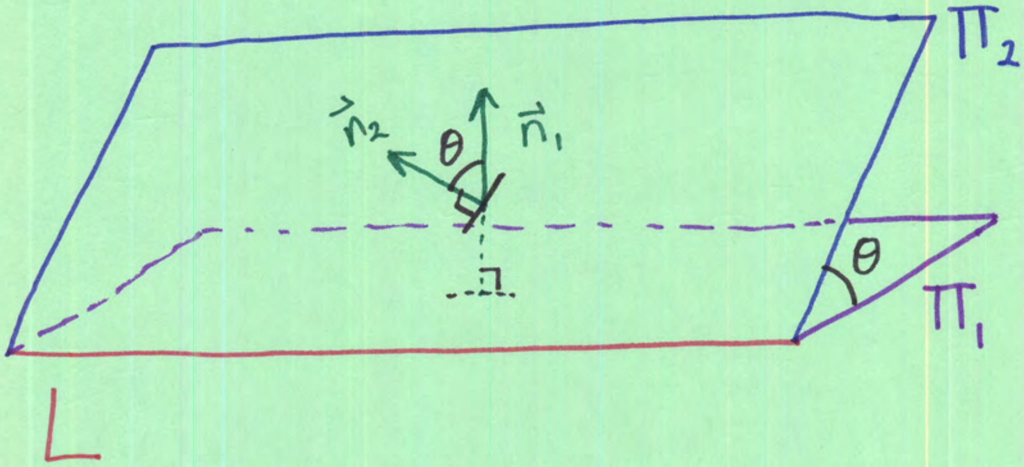
$$x+y+z = (3+3t)+(t)+(-2+4t)$$
$$= 1+8t = 2 \Rightarrow t = \frac{1}{8}$$

So, they do intersect, and the point of intersection is $(x,y,z) = \left(3+3\left(\frac{1}{8}\right), \frac{1}{8}, -2+4\left(\frac{1}{8}\right)\right)$

$$= \left(\frac{27}{8}, \frac{1}{8}, \frac{-3}{2}\right)$$

$\diamond$

How, now, about 2 planes? It's possible they're parallel (to check this, check if their normal vectors are parallel). More likely, though, they'll intersect. As you can probably see, the don't intersect in a point, but in a line!




Ex: Do the planes $2x-3y+4z=5$ and $x+6y+4z=3$ intersect? If so, what is the angle of their intersection? also, give an equation for their line of intersection.

Sol: The normal vectors of the planes are

$$\vec{n}_1 = \langle 2,-3,4\rangle \ \& \ \vec{n}_2 = \langle 1,6,4\rangle$$

which can easily be seen to not be parallel since one is not a multiple of the other. So the planes are not parallel, thus they intersect. The angle of intersection is the same as the angle between their normal vectors:

$$\theta = \arccos\left(\frac{\vec{n}_1\cdot\vec{n}_2}{\|\vec{n}_1\|\,\|\vec{n}_2\|}\right) = \arccos\left(\frac{(2)(1)+(-3)(6)+(4)(4)}{(\sqrt{4+9+16})(\sqrt{1+36+16})}\right) = \arccos(0)$$

$$= \frac{\pi}{2}$$

(This actually means the planes are perpendicular!)
Now, for the line of intersection, we need a point and a direction vector. Let's start with the direction. The line lies in both planes, so its direction vector $\vec{v}$ must be perpendicular to both $\vec{n}_1$ & $\vec{n}_2$ since its parallel to both planes. We have a trick for creating a vector orthogonal to two given vectors: the cross product.

$$\vec{v} \parallel \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 1 & 6 & 4 \end{vmatrix} = \langle -12-24, -(8-4), 12-(-3) \rangle$$

$$= \langle -36, -4, 15 \rangle$$

We may as well choose $\vec{v} = \vec{n}_1 \times \vec{n}_2$. Now, for a point on the line, we just need to find a point on both planes, that is, a solution to both $2x-3y+4z=5$ and $x+6y+4z=3$. We have 2 equations and 3 variables, so we'll have to choose a value for one of them, say $z=0$. Then, we need to solve the system:

$$\begin{cases} 2x-3y=5 \quad (1) \\ x+6y=3 \quad (2) \end{cases}$$

$(2)+2(1)$: $5x = 13 \Rightarrow x = \frac{13}{5}$

Plug into $(2)$: $6y = 3 - \frac{13}{5} = \frac{2}{5} \Rightarrow y = \frac{1}{15}$

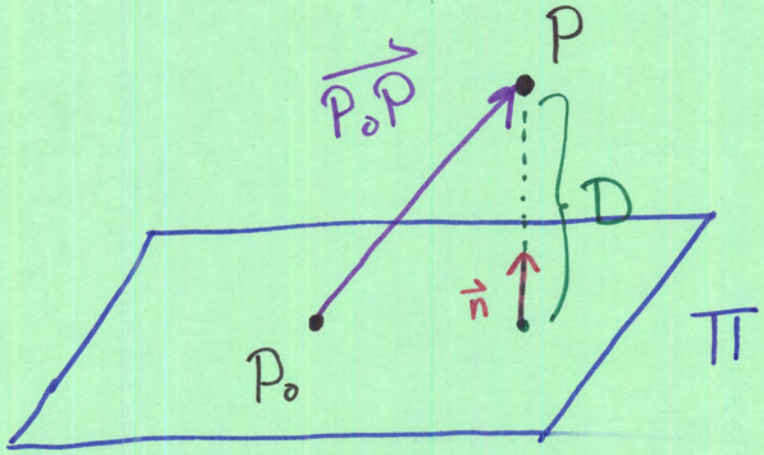
So, a point on the line is $(\frac{13}{5}, \frac{1}{15}, 0)$.

The symmetric equations for this line are then

$$\frac{x-\frac{13}{5}}{-36} = \frac{y-\frac{1}{15}}{-4} = \frac{z}{15}$$

$\square$

Consider the following situation:




We're given a plane $\Pi$ and a point P. How can we find the distance, D, from the plane to the point? First, we know that the shortest path from the plane to point is a straight line perpendicular to the plane, that is a line in the direction of $\vec{n}$, the normal vector to $\Pi$. Notice that if we take some point $P_0$ on $\Pi$ and connect it to P, we get a vector connecting $\Pi$ to P, and, moreover, if we project $\overrightarrow{P_0P_1}$ onto $\vec{n}$, we get a vector $\perp$ to $\Pi$ which starts on $\Pi$ and ends at $P_1$. The length of this vector, then, is precisely $D$, i.e.,

$$\boxed{D = \|\text{proj}_{\vec{n}}\, \overrightarrow{P_0P_1}\| = |\text{comp}_{\vec{n}}\, \overrightarrow{P_0P_1}|}$$

If $\vec{n} = \langle a, b, c\rangle$, $P_0 = (x_0, y_0, z_0)$, & $P_1 = (x_1, y_1, z_1)$, then

$$D = |\text{comp}_{\vec{n}}\, \overrightarrow{P_0P_1}| = \frac{|\vec{n} \cdot \overrightarrow{P_0P_1}|}{\|\vec{n}\|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}.$$

If the plane is written as $ax + by + cz + d = 0$, then

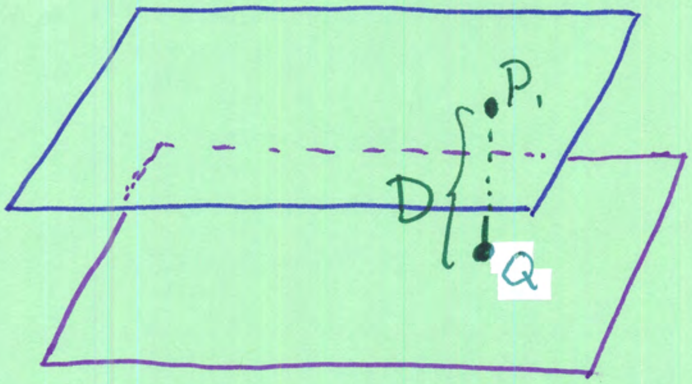
$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Let's see how this can be used to answer a related question.

<u>Ex</u>: Find the distance between the parallel planes

$x - 4y + 2z = 0$ and $2x - 8y + 4z = -1$.

Sol: Our situation looks as follows:




If we forget everything except $P_1$ from the blue plane, we've reduced the problem to the distance between a point and a plane. First, we need to find a $P_1$ (it doesn't matter which plane $P_1$ is on, as long as $P_0$ is on the other one). Let's take $P_1$ on the second plane. Any point works, so the easiest way to get at one is to make 2 components equal to zero, e.g., take $P_1 = (-\frac{1}{2}, 0, 0)$. A point on the other plane is $P_0 = (0,0,0)$.

A normal vector to the planes is $\vec{n} = \langle 1, -4, 2 \rangle$, so

$$D = |\mathrm{comp}_{\vec{n}} \overrightarrow{P_0P_1}| = \frac{|\vec{n} \cdot \overrightarrow{P_0P_1}|}{\|\vec{n}\|} = \frac{|(1)(-\frac{1}{2}) + (-4)(0) + (2)(0)|}{\sqrt{1+16+4}} = \frac{\frac{1}{2}}{\sqrt{21}} = \frac{1}{2\sqrt{21}}$$

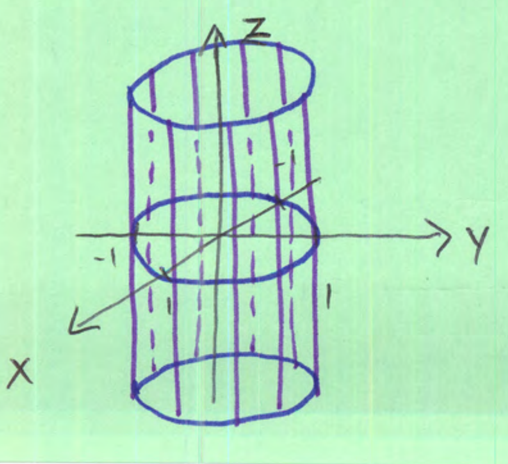
$\Diamond$

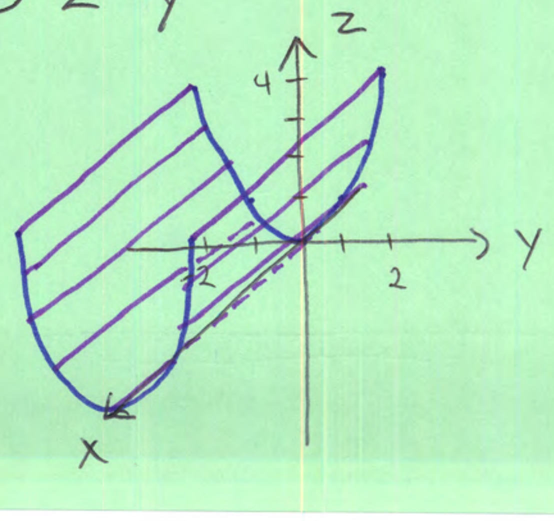
# 12.6 - Cylinders and Quadric Surfaces

The point of covering this section is to get you familiar with some of the surfaces we will be working with in the future. We will not cover the material as in-depth as the book does.

Cylinders: A cylinder is a surface consisting of all lines (called rulings) parallel to a given line line and which pass through a given plane curve. In a lot less fancy words, this essentially means you can roll up a piece of paper to look like the surface (well, a piece of it anyway).

Ex: (a) $x^2+y^2=1$ (b) $z=y^2$



(The blue curves are the aforementioned plane curves and the purple lines the rulings.)

## Quadric Surfaces

A quadric surface is simply the graph of a second degree polynomial in x, y, and z. The most general such equation is

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0.$$

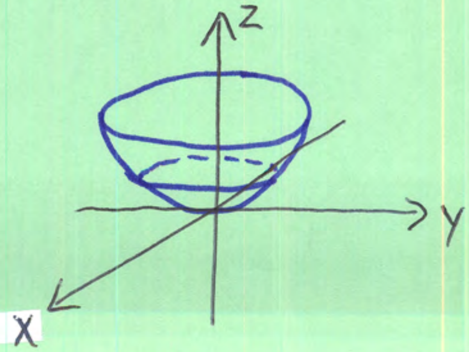
However, through rotations and translations, they can all be made to look like

$$Ax^2+By^2+Cz^2+J=0 \quad \text{or} \quad Ax^2+By^2+Iz=0$$

Let's look at the ones of these which will show up for us later:

① (Elliptic) Paraboloids : $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ (Elliptic if $a \neq b$)

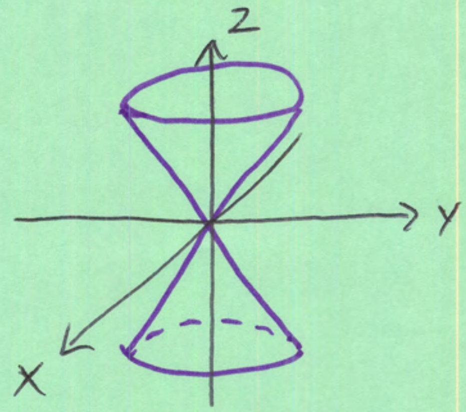
Ex: $z=2x^2+y^2$




"Looks like a bowl."

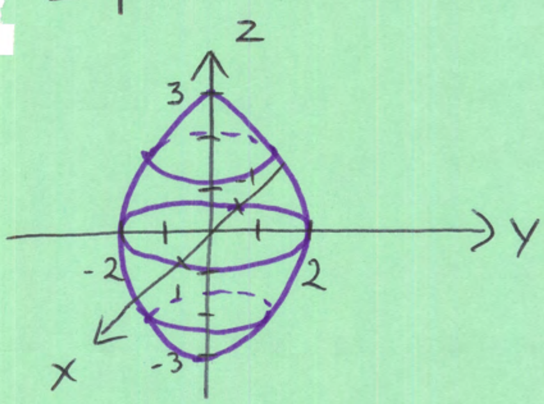
② <u>Cones</u>: $\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

<u>Ex</u>: $z^2 = x^2 + y^2$




③ <u>Ellipsoids</u>: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ (if $a=b=c$, this is a sphere)

<u>Ex</u>: $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$




"Looks like a rugby ball."